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VICTOR AUGUST HOERSCH
B.A., State University of Iowa, 1915
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THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN PHYSICS, IN THE GRADUATE COLLEGE OF THE STATE UNIVERSITY OF IOWA, AUGUST, 1922

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By V. A. Hoersch

Abstract

Simple harmonic vibrations satisfying the equation

$$\phi = (kr)^{-\frac{1}{2}} J_{m+\frac{1}{2}}(kr) P_m(\cos\theta)\cos\sigma t$$

are studied. In this equation, ϕ is the velocity potential, J and P denote Bessel and Legendre functions respectively, and r and θ are polar coordinates. The parameter m specifying the orders of the Bessel and Legendre functions is determined so that the vibrations satisfy the boundary conditions for a conical horn. This is possible by means of a *new expansion for* $P_m(x)$ which is herein developed. With the assumption of a loop at the opening of the horn and by the aid of an asymptotic expansion for $J_{m+\frac{1}{2}}(z)$, numerical values are computed, for horns of various angles (2° to 30°) and for two types of vibration, of the ratios n/n_0 of several frequencies to the fundamental frequency n_0, and of the ratios λ/d of the corresponding wave-lengths to the diameter of the horn at the opening. The nature of these two types of vibration is indicated by figures.

A CONICAL horn, used as a receiver of sound, gives greatest amplification when of such length that one of its free periods of vibration coincides with the period of the incident sound. Experiments of G. W. Stewart[1] show that for a given type of resonance, fundamental or any overtone, the amplification is greatest for an angle termed the optimum angle. The optimum angle is greatest for fundamental resonance and decreases for each successive overtone. The theory of vibrations within horns has heretofore dealt exclusively with radial vibrations, that is, waves concentric with the vertex of the horn. In this case the inwardly directed waves come to a focus at the vertex and the reflected waves proceed outward in spherical wave-fronts. It has been suggested[1] that the optimum angle effect might be due to the presence of other types of vibration within the horn.[2] A study of such vibrations is undertaken in the present article.

In the case of resonance to harmonic waves of one frequency we may, in the case of a conical horn whose walls extend to infinity, put the velocity potential equal to $\phi \cos \sigma t$ in which ϕ is independent of the time t. We may take for ϕ a value also suitable for the vibrations within a rigid spherical enclosure, viz.[3]

$$\phi = (\mathrm{kr})^{-\frac{1}{2}} J_{m+\frac{1}{2}}(\mathrm{kr}) \; P_m(\cos\theta) \qquad (1)$$

* Another part of the doctor's thesis was published in the Physical Review, pp. 225–229, No. 2, February 1925 under the title "Theory of the Optimum Angle in a Receiving Conical Horn."

[1] G. W. Stewart, Phys. Rev. **16,** 310 (1920).

[2] A different cause is claimed by the author in the following article.

[3] Lamb, Hydrodynamics (4th ed. 1916), art. 292, p. 497.

in which $J_{m+\frac{1}{2}}$ and P_m denote Bessel and Legendre functions of orders $m+\frac{1}{2}$ and m respectively and $k=\sigma/c$ (c = velocity of sound); r and θ are polar coordinates with origin at the vertex of the horn and with polar axis the axis of the horn. Eq. (1) includes radial vibrations as a special case when $m=0$ since $P_0(\cos\theta)=1$ and ϕ is no longer a function of θ. If m is not zero in Eq. (1), the following condition insuring the vanishing of flow across the boundary must be satisfied:

$$\partial\phi/\partial\theta \propto P'_m(\cos\theta_0) = 0 \tag{2}$$

where $\theta=\theta_0$ at the rigid wall of the horn. It will appear from the succeeding development that this restricts the parameter m in Eq. (1) to an infinite set of discrete values. It is our purpose to find this set of values but preparatory thereto we must obtain a convenient development of $P_m(x)$ as a function of m.

We may start with the expansion[4]

$$P_m(x)\equiv 1+\sum_{r=1}^{\infty}[(-1)^r/r!^2](m-r+1)(m-r+2)\cdots(m+r)\,[\tfrac{1}{2}(1-x)]^r. \tag{3}$$

Let the coefficient of $[\frac{1}{2}(1-x)]^r$ be

$$[(-1)^r/r!^2]\sum_{s=0}^{2r-1}C_{s,r}m^{2r-s}. \tag{4}$$

Then Eq. (3) becomes

$$P_m(x)\equiv 1+\sum_{r=1}^{\infty}\sum_{s=0}^{2r-1}[(-1)^r/r!^2]C_{s,r}m^{2r-s}[\tfrac{1}{2}(1-x)]^r. \tag{5}$$

The series $P_m(x)$ is dominated by the positive series

$$S_m(x)\equiv 1+\sum_{r=1}^{\infty}(r!)^{-2}(m+r-1)(m+r-2)\cdots(m+1)m(m+1)\cdots(m+r)\,[\tfrac{1}{2}(1-x)]^r \tag{6}$$

regarded as a double series in m and $\frac{1}{2}(1-x)$. The series $S_m(x)$ by the use of Cauchy's ratio test is seen to converge for $-1<x<3$ for all values of m. Hence the series $P_m(x)$ converges absolutely under the same conditions.[5]

By comparing the term in $[\frac{1}{2}(1-x)]^{r+1}$ and the term in $[\frac{1}{2}(1-x)]^r$ in Eq. (3) we have

$$\sum_{s=0}^{2r-1}C_{s,r+1}m^{2r-s+2}\equiv(m-r)\,(m+r+1)\sum_{s=0}^{2r-1}C_{s,r}m^{2r-s}. \tag{7}$$

[4] Whittaker and Watson, Modern Analysis (1920), art. 15.22, p. 311.

[5] This method of proof was suggested by Dr. E. W. Chittenden, Department of Mathematics, University of Iowa.

After expanding the right member and arranging it in descending powers of m, we find the coefficient of m^{2r-s+2} in this member to be $C_{s,r}+C_{s-1,r}-r(r+1)C_{s-2,r}$. Since the coefficients of like powers of m on the two sides of Eq. (7) must be equal, we have

$$C_{s,r+1}-C_{s,r}=C_{s-1,r}-r(r+1)C_{s-2r} \tag{8}$$

A comparison of Eqs. (3) and (5) shows that $C_{0,r}=1$ and $C_{1,r}=r$ for every r. From Eq. (8) we thus obtain

$$C_{2,r+1}-C_{2,r}=-r^2 \tag{9}$$

whence
$$C_{2,r}=-\sum_{1}^{r-1} k^2=-\tfrac{1}{6}r(r-1)(2r-1). \tag{10}$$

The right member of Eq. (8) for $s=3$ is now known and hence

$$\begin{aligned} C_{3,r}&=-\sum_{1}^{r-1}[C_{2,k}-k(k+1)C_{1,k}] \\ &=\tfrac{1}{6}\sum_{1}^{r-1} k(8k^2+3k+1) \\ &=-\tfrac{1}{6}r^2(r-1)(2r-1). \end{aligned} \tag{11}$$

Similarly successive applications of Eq. (8) give

$$C_{4,r}=\tfrac{1}{360}r(r-1)(r-2)(20r^3-36r^2+7r+3) \tag{12}$$

and
$$C_{5,r}=\tfrac{1}{360}r^2(r-1)(r-2)(20r^3-36r^2+7r+3). \tag{13}$$

Thus as many of the constants $C_{s,r}$ of Eq. (5) as may be desired may be found.

We proceed now to show how $P_m(x)$ can be made to depend on the Bessel functions. Interchanging the order of summation and substituting $\frac{1}{2}m^2(1-x)=(\frac{1}{2}y)^2$, Eq. (5) becomes

$$P_m(x)\equiv 1+\sum_{s=0}^{\infty}(1/m^s)\sum_{r=1}^{\infty}[(-1)^r/r!^2]C_{s,r}(\tfrac{1}{2}y)^{2r}. \tag{14}$$

Let the coefficient of $1/m^s$ in Eq. (14) be $Y_s(y)$. Substituting the values of $C_{s,r}$ just found in the expressions for $Y_s(y)$ and comparing with the series expansions of Bessel functions, we have

$$Y_0(y) = J_0(y) \tag{15}$$

$$Y_1(y)=-\tfrac{1}{2}yJ_1(y) \tag{16}$$

$$Y_2(y)=-\tfrac{1}{6}(\tfrac{1}{2}y)^2J_0(y)+[-\tfrac{1}{3}(\tfrac{1}{2}y)^3+\tfrac{1}{6}(\tfrac{1}{2}y)]J_1(y) \tag{17}$$

$$Y_3(y)=-\tfrac{1}{3}(\tfrac{1}{2}y)^4J_0(y)-\tfrac{1}{6}(\tfrac{1}{2}y)^3J_1(y) \tag{18}$$

$$Y_4(y)=\left[-\tfrac{1}{18}(\tfrac{1}{2}y)^6-\tfrac{1}{40}(\tfrac{1}{2}y)^4+\tfrac{1}{60}(\tfrac{1}{2}y)^2\right]J_0(y)$$
$$+\left[-\tfrac{1}{15}(\tfrac{1}{2}y)^5+\tfrac{1}{30}(\tfrac{1}{2}y)^3-\tfrac{1}{60}(\tfrac{1}{2}y)\right]J_1(y) \tag{19}$$

$$Y_5(y)=\left[-\tfrac{7}{30}(\tfrac{1}{2}y)^6-\tfrac{1}{60}(\tfrac{1}{2}y)^4\right]J_0(y)$$
$$+\left[\tfrac{1}{18}(\tfrac{1}{2}y)^7-\tfrac{13}{120}(\tfrac{1}{2}y)^5+\tfrac{1}{60}(\tfrac{1}{2}y)^3\right]J_1(y). \tag{20}$$

In general $Y_s(y)$ may be expressed linearly in $J_0(y)$ and $J_1(y)$, the coefficients being rational integral functions of y. The series $P_m(x)=1+\Sigma_0^\infty m^{-s}Y_s(y)$ is convergent for $y/2m$ less in absolute value than 1. A geometrical series may be found which dominates this series and by its means a limit of error may be found for the residue after the term in m^{-5}. If R is the absolute value of this residue, we find that $R<.0067$ for $y=4$ and $m=10$, $R<.000015$ for $y=4$ and $m=20$, and $R<.0067$ for $y=7.2$ and $m=40$. These limits for R are, however, not close.

We are now ready to utilize the series just obtained for $P_m(x)$ in Eq. (2), which becomes, if $x=\cos\theta_0$,

$$dP_m(x)/dy=\sum_0^\infty m^{-s}dY_s(y)/dy=0, \tag{21}$$

or substituting $m^{-1}=2\xi/y$,

$$\sum_0^\infty \xi^s(\tfrac{1}{2}y)^{-s}dY_s(y)/dy=0. \tag{22}$$

Expanding $(\frac{1}{2}y)^{-s}dY_s(y)/dy$ as a Taylor series in powers of $\eta=y-y_1$ where y_1 is a root of $J_1(y)=0$, and substituting in Eq. (22), we have a double series in ξ and η equated to zero. Because of its length this series is not here reproduced. By a reversion of this series, using undetermined coefficients, we obtain

$$\eta=-\xi+(-\tfrac{1}{6}y_1+\tfrac{1}{2}y_1^{-1})\xi^2+(-\tfrac{17}{360}y_1+\tfrac{13}{60}y_1^{-1}-\tfrac{1}{8}y_1^{-3})\xi^4$$
$$+(\tfrac{2998}{3780}y_1^2-\tfrac{1}{12}+\tfrac{1}{4}y_1^{-2}-\tfrac{1}{4}y_1^{-4})\xi^5+\cdots \tag{23}$$

where $\xi=\sin(\frac{1}{2}\theta_0)$.

Since $\xi m=\frac{1}{2}(y_1+\eta)$ defines a value of m satisfying Eq. (2) for each root y_1 of $J_1(y)=0$, we have an infinite sequence of values of m which, substituted in Eq. (1), define different modes of vibration for a given horn. These modes of vibration will be called I, II, etc., corresponding to the first, second, etc., roots y_1 of $J_1(y)=0$. For types I and II, $y_1=3.83171$ and 7.01559 respectively, and by means of Eq. (23) we have

$$\xi m=1.91585-\tfrac{1}{2}\xi-.25406\xi^2-.06331\xi^4+.54694\xi^5+\cdots \tag{24}$$

$$\xi m=3.50779-\tfrac{1}{2}\xi-.54900\xi^2-.15038\xi^4+1.90742\xi^5+\cdots \tag{25}$$

Let us now turn from the horn of infinite length to one of finite length. The vibrations obtained for the former will also be valid for the latter if the proper condition is imposed at the opening. In Eq. (1) the parameter m has been chosen but we still have the parameter k at our disposal. If we assume that the inertia of the medium outside the horn is negligible, we may take $p=0$ at the opening of the horn, where p is the excess of pressure over the mean pressure. Since

$$pe^{i\sigma t}=\rho\partial(\phi e^{i\sigma t})/\partial t=i\sigma\rho\phi e^{i\sigma t}, \tag{26}$$

where ρ is the mean density of the medium, $p=0$ entails $\phi=0$ at the opening. This condition will be fulfilled if we take k in Eq. (1) such that

$$J_{m+\frac{1}{2}}(kl)=0 \tag{27}$$

where l is the length of the horn. Now $J_{m+\frac{1}{2}}(z)=0$ has an infinite sequence of roots z_1, z_2, z_3, etc., and kl may be taken equal to any one of them. Corresponding to these roots we have vibrations I_1, I_2, I_3, etc., respectively, all of type I; similarly II_1, II_2, etc., all of type II, etc.

Numerical computations were effected for vibrations I_1, I_2, I_3, II_1, II_2, and II_3. The results are shown in Table I and Fig. 1. For each

TABLE I

Frequencies and wave-lengths of non-radial vibrations as functions of the angle of the horn

	θ_0	n_1/n_0	n_2/n_0	n_3/n_0	λ_1/d	λ_2/d	λ_3/d
TYPE I	2°	37.85	40.11	41.99	.7572	.7143	.6823
	4°	19.80	21.66	23.25	.7238	.6617	.6166
	6°	13.71	15.38	16.83	.6977	.6221	.5685
	8°	10.63	12.18	13.55	.6760	.5895	.5302
	10°	8.763	10.24	11.54	.6570	.5624	.4987
	15°	6.242	7.595	8.811	.6189	.5087	.4386
	20°	4.96	6.24	7.40	.589	.468	.395
	25°	4.18	5.41	6.54	.566	.437	.362
	30°	3.65	4.70	5.96	.548	.425	.336
TYPE II	2°	67.51	70.19	72.48	.4244	.4082	.3954
	4°	34.79	36.99	38.87	.4120	.3876	.3688
	6°	23.80	25.77	27.44	.4019	.3712	.3489
	8°	18.26	20.08	21.63	.3934	.3578	.3322
	10°	14.92	16.63	18.10	.3860	.3463	.3181
	15°	10.41	11.96	13.31	.3711	.3230	.2902
	20°	8.13	9.58	10.82	.359	.305	.270
	25°	6.75	8.13	9.36	.315	.291	.253
	30°	5.82	7.15	8.35	.344	.280	.239

value of θ_0 values of m were obtained from Eqs. (24) and (25) for vibrations of types I and II respectively. The first three roots z_1, z_2, z_3 of $J_{m+\frac{1}{2}}(z)=0$ were found by successive approximation, using an expansion

for $J_{m+\frac{1}{2}}(z)$ due to Nicholson.[6] Then k was taken equal to z_i/l for $i=1$, 2, 3. The corresponding frequencies n were obtained by multiplying each k by $c/2\pi$, and the corresponding wave-lengths λ were obtained from the relation $\lambda=2\pi/k$. Table I gives the ratio of the frequencies to n_0, the fundamental frequency of the horn, and the ratio of the wave-lengths to d, the diameter of the horn opening. In Fig. 1 the latter ratio is plotted against θ_0. As the angle θ_0 approaches zero, the cone approaches more and more closely to a cylinder and all vibrations of type I approach a single cylindrical vibration; similarly for vibrations of type II, etc.

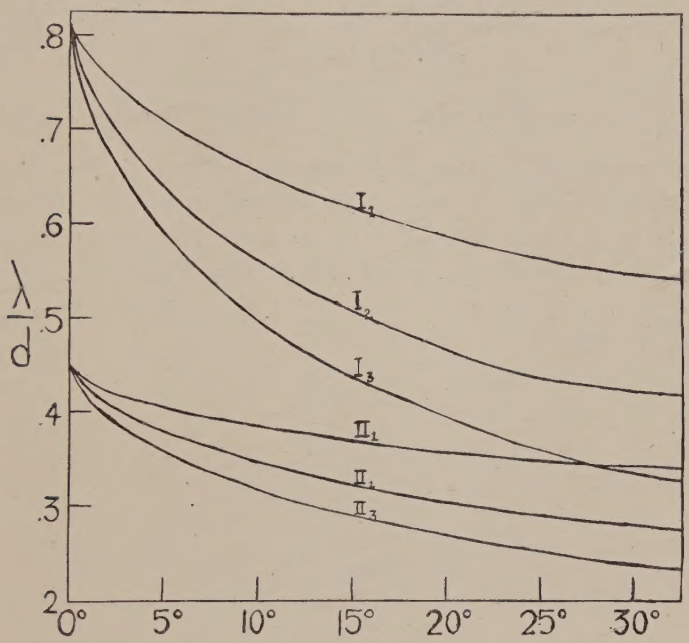

Fig. 1. Wave-lengths of new vibrations as function of angle of horn θ_0.

Figs. 2 and 3 show the traces of the equipotential surfaces on a plane through the axis of the horn. The equations of these curves in polar coordinates are obtained by assigning to ϕ in Eq. (1) successive constant values which are the terms of an arithmetic series. The velocity of the medium is, of course, normal to the equipotential surfaces and directly proportional to the gradient. It is seen from these figures that the vibrations may be roughly characterized as combined radial and transverse vibrations. The present investigation does not answer the question of the strength of the resonating response for any of the types of vibration studied. As always in acoustics, the separate vibrations

[6] Nicholson, Phil. Mag. **14**, 697 (1907).

here studied may be present simultaneously in any combination among themselves or with the classical vibrations of radial type.

In conclusion the author wishes to express his appreciation to members of the Department of Physics of the State University of Iowa for

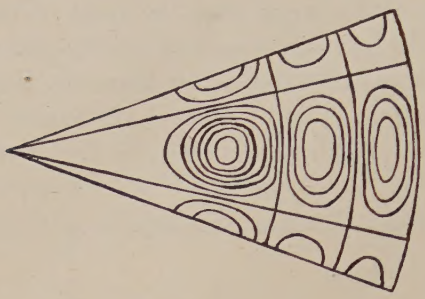

Fig. 2. Vibration of type I.

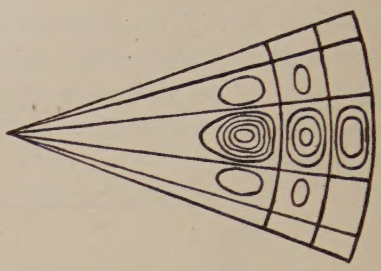

Fig. 3. Vibration of type II.

their assistance and interest, and especially to Professor G. W. Stewart under whose direction this investigation was carried on. The author is also grateful to Dr. E. W. Chittenden for his help in the proof of the convergence of the double series representing $P_m(x)$.

UNIVERSITY OF IOWA,
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